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THE STABILITY OF A FAMILY OF NON-TRIVIAL EQUILIBRIUM ORIENTATIONS TO THE ATTRACTING CENTRE OF A GYROSTAT WITH AN ELASTIC ROD†

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The motion of a system consisting of a gyrostat and an elastic rod in a circular Kepler orbit in a central Newtonian force field is considered in a restricted formulation. The gyrostat is treated as a rigid body in which there is a dynamically rotating flywheel and a statically counterbalanced flywheel. The uniform elastic rod, which is rectilinear in the undeformed state, is rigidly fixed to the gyrostat housing at one end. The axis of the undeformed rod is arbitrarily located in the principal plane of inertia of the gyrostat. The relative displacements of the points of the system as a result of a small deformation of its elastic link are represented in the form of an infinite series of its expansion (without its *a priori* truncation) in a specified system of functions, which depend on the spatial coordinates, with unknown time-dependent coefficients. The orientation of the system for an attracting centre is defined by indicating the position with respect to the associated system of coordinates of the unit vectors of the normal to the plane of the orbit and the radius vector or transversal of the orbit at the centre of mass of the system. Here, these two unit vectors are located in the principal central plane of inertia of the gyrostat, containing the axis of the undeformed rod. The deformations of the rod, which naturally depend on the orientation and the gyrostatic moment which ensures equilibrium of the chosen orientation (non-trivial equilibrium since, in this case, generally speaking, the rod is deformed), and its stability in the Lyapunov sense are determined for the two single parameter families of uniaxial orientations of the system to an attracting centre which have been separated out in this way.

The problem of the steady motions of a gyrostat [1, 2], which is treated here as a rigid body with a rotating statically counterbalanced flywheel and a dynamically rotating flywheel positioned in it is customarily separated into a direct and an inverse problem. In the direct problem (the problem of analysis) it is necessary to find the steady motions (the equilibria, in particular) for a given gyrostatic moment of the system. In the inverse formulation (the problem of synthesis) the gyrostatic moment which ensures the chosen steady motion of the system is sought.

For a gyrostat in a circular Kepler orbit in the direct formulation, an analytical solution of the problem is only known for the case when the vector of the gyrostatic moment of the system is located in a particular principal centre plane (see [3], for example). In the general case of the location of the flywheel, results are available based on numerical calculations (see [4, 5], for example) but this important and extensive class of investigations of the problem is not discussed any further. Results are more abundant for the inverse formulation of the problem [5–8]. In the last of these papers, in particular, the accepted classification of the relative equilibria of a gyrostat in a circular orbit is presented. It should be noted that the solution of the problem of the possibility, because of the choice of the gyrostatic moment of the system, of ensuring a relative equilibrium for which an arbitrarily specified axis, fixed in the main body of the gyrostat relative to the orbital system of coordinates, coincides, in the case of an arbitrary form of the determined axis, with a fixed axis relative to the orbital system of coordinates, is successfully reduced to finding the real root of the corresponding fourth-order algebraic equation. However, the conditions, relating the parameters of the system and which guarantee the existence of such a solution have not been given in the literature.

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Research which has been published on the above-mentioned problems for a gyrostat with an elastic element is not very extensive and is concerned with the trivial equilibria of a system when its elastic element has not been deformed (see [9], for example).

1. FORMULATION OF THE PROBLEM. THE INTEGRALS OF MOTION

We will consider, in a restricted formulation [1 = 0], the motion of a mechanical system, consisting of a gyrostat and an elastic rod, in a circular Kepler orbit in a central Newtonian force field. The gyrostat is treated as a rigid body in which there is a dynamically rotating flywheel and a statically counterbalanced flywheel. A uniform elastic rod, which is rectilinear in the undeformed state, is rigidly fixed in the gyrostat housing at one end. The axis of the undeformed rod is arbitrarily located in the principal central plane of inertia of the gyrostat. As the system moves, its instantaneous centre of mass is displaced in a circular Kepler orbit around the centre of attraction and the rod undergoes small spatial flexural oscillations.

The purpose of this paper is to show that, in the case of this system, non-trivial families of relative equilibria of the second and third classes [8] exist, and to point out the conditions for their Lyapunov stability.

Only right-handed systems of Cartesian coordinates are introduced to describe the motion. The system of coordinates Oy_k ($k = 1, 2, 3$) is introduced with a pole O at the instantaneous centre of mass and the unit vectors of the axes $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ respectively; the unit vector $\boldsymbol{\beta}$ is directed along the normal to the orbital plane and $\boldsymbol{\gamma}$ is directed along the radius vector of the instantaneous centre of mass relative to the attracting centre. The constant angular velocity vector of the rotation of the orbital system of coordinates with respect to inertial space $\boldsymbol{\omega} = \omega\boldsymbol{\beta}$, $\omega > 0$, R is the radius of the circular orbit of the motion of the centre of mass O, L is the characteristic size of the system and m is its mass. The system of coordinates O_1x_k with the unit vectors of the axes $\mathbf{i}_k$ ($k = 1, 2, 3$) is rigidly fixed to the housing of the gyrostat, O_1 is the centre of mass of the undeformed system while the coordinate axes coincide with the principal central axes of the gyrostat, and Ω is the angular velocity vector of the trihedron O_1x_k with respect to Oy_k.

Suppose the axis of the elastic rod, which is rectilinear in the undeformed state and, for simplicity, of constant circular cross-section and unit length, is located in the plane $O_1x_2x_3$, ρ is the mass per unit length of the rod, a is the distance from the point O_1 to the point where the rod is fastened, and the parameter $s \in [0, 1]$ defines a point on the axis of the rod. We will assume that, as the rod moves, it suffers small spatial flexural deformations in accordance with Kirchhoff's hypotheses: the cross-sections of the rod are not deformed, and their twisting and the change in the normal of the transverse cross-section relative to the normal of the same cross-section in the undeformed position of the rod are neglected.

The points of the gyrostat occupy a bounded domain υ_1 while the points of the undeformed elastic link occupy the bounded domain υ_2, Γ is the common boundary of the domains, $\dim\Gamma \neq 0$, and $\upsilon = \upsilon_1 + \upsilon_2$ [11].

In order to describe the deformations of the elastic link of the system, we use a local system of coordinates with unit vectors $\{\mathbf{f}_k\}$; the unit vector $\mathbf{f}_3$ is located along the axis of the undeformed rod which passes through the point O_1 and is directed from it. The radius vector of an arbitrary point of the rod, which, prior to deformation, is defined with respect to the point O_1 by the vector $\mathbf{r}$, will be defined after deformation with respect to the instantaneous centre of mass of the system O by the expression $(\mathbf{r} + \mathbf{u}(t, s) - \mathbf{r}_0)$, where $\mathbf{u}(t, s)$ is the vector of the elastic displacement of the points of the rod axis and $\mathbf{r}_0 = m^{-1}\int_0^1\rho\mathbf{u}(t, s)ds$ is the radius vector of the point O with respect to the point O_1. We shall henceforth neglect the quantity $\mathbf{r}_0$, that is, it is assumed that the points O_1 and O coincide.

We will now formulate the assumptions employed in this paper.

1. We will represent the vector of the elastic displacement of the rod axis as follows:

$$\mathbf{u}(t, s) = \sum_{p=0}^{\infty}(q_p^{(1)}\chi_p^{(1)}(s)\mathbf{f}_1 + q_p^{(2)}\chi_p^{(2)}(s)\mathbf{f}_2) = \sum_{n=1}^{\infty}\tilde{q}_n(t)\tilde{\boldsymbol{\varphi}}_n(s) \quad (1.1)$$

where

$$\tilde{q}_{2p+i}(t) = q_p^{(i)}, \quad \tilde{\boldsymbol{\varphi}}_{2p+i}(s) = \chi_p^{(i)}(s)\mathbf{f}_i; \quad p = 0, 1, \ldots; \quad i = 1, 2$$

Note that the generalized coordinates $\tilde{q}_{2k-1}(t)$ define elastic displacements along the axis $\mathbf{f}_1$ while $\tilde{q}_{2k}(t)$ define elastic displacements along the axis $\mathbf{f}_2$ ($k = 1, 2, \ldots$) lying in the plane $O_1x_2x_3$. Functions of the parameter s satisfying the corresponding boundary conditions (one end of the rod is rigidly clamped and the other is free) are represented and follows:

$$\chi_{n-1}(s) = \chi_{n-1}^{(1)}(s) = \chi_{n-1}^{(2)}(s) = ((\operatorname{sh}\beta_n + \sin\beta_n)(\operatorname{ch}\beta_n s - \cos\beta_n s) - \\ -(\operatorname{ch}\beta_n + \cos\beta_n)(\operatorname{sh}\beta_n s - \sin\beta_n s))/(\sin\beta_n \operatorname{ch}\beta_n - \cos\beta_n \operatorname{sh}\beta_n) \tag{1.2}$$

The quantities β_n ($n = 1, 2, \ldots$) are the roots of the equation $\cos\beta\,\operatorname{ch}\beta + 1 = 0$ for which the functions are normalized such that

$$\int_0^1 \rho\chi_n(s)\chi_p(s)ds = M_n\delta_{np}, \quad M_n = 1$$

2. The potential energy of small elastic deformations is defined by the expression

$$\Pi = \frac{1}{2}\sum_{n,p=1}^{\infty} \tilde{c}_{np}\tilde{q}_n\tilde{q}_p, \quad \tilde{c}_{np} = \Lambda_n^2 M_n\delta_{np}, \quad \Lambda_n^2 = \frac{EI\beta_n^4}{\rho} \tag{1.3}$$

where $\Lambda_n(M_n)$ is the frequency (reduced mass) corresponding to the mode $\chi_{n-1}(s)$, and EI is the stiffness of the rod. It is clear that $\Lambda_1 < \Lambda_2 < \ldots$ and it is also natural to assume that the potential energy of the elastic deformations remains bounded. If the new variables $q_n(t) \equiv (\tilde{c}_{nn})^{1/2}\tilde{q}_n(t)$ and, correspondingly, $\varphi_n(s) \equiv (\tilde{c}_{nn})^{-1/2}\tilde{\varphi}_n(s)$ are introduced, we conclude from the boundedness of the energy that $q(t) \equiv (q_1, q_2, \ldots)$ belongs to the Hilbert space l_2 of infinite sequences which are bounded with respect to the norm

$$\|q\| = \left(\sum_{n=1}^{\infty}|q_n|^2\right)^{1/2}$$

It should be noted that $\Lambda_1 > 1$ and, as has been shown in [14], $\{\Lambda_n^{-1}\} \in l_1$. Consequently, $\{\Lambda_n^{-2}\} \in l_2$.

3. Neglecting quantities of the order of $(L/R)^3$ and higher, we use the following approximate expression for the potential energy of the gravitational forces

$$\Pi_g = -\frac{\mu m}{R} + \frac{1}{2}\omega^2(3\boldsymbol{\gamma}\mathbf{J}\boldsymbol{\gamma} - \operatorname{tr}\mathbf{J}) \tag{1.4}$$

Here μ is the product of the gravitational constant and the mass of the attracting centre and $\mathbf{J}$ is the inertia tensor of the system with respect to the instantaneous centre of mass.

In accordance with representation (1.1)

$$\mathbf{J}(\mathbf{q}) \equiv \int_v((\mathbf{r}+\mathbf{u})^2E - (\mathbf{r}+\mathbf{u}):(\mathbf{r}+\mathbf{u}))dm = \\ = \mathbf{J}_0 + \sum_{n=1}^{\infty} q_n(t)\mathbf{J}_n + \sum_{n,p=1}^{\infty} q_nq_p\mathbf{J}_{np} \tag{1.5}$$

where $\mathbf{J}_0$ is the inertial tensor of the undeformed system with respect to the point O, E is a 3×3 unit matrix and a colon denotes a dyadic product of vectors. We will represent the matrices of the components of the tensors $\mathbf{J}_0$, $\mathbf{J}_n$, $\mathbf{J}_{np}$ in the local system of coordinates $\{\mathbf{f}_k\}$

$$[\mathbf{J}_0]_F = \begin{Vmatrix} J_0^{11} & 0 & 0 \\ 0 & J_0^{22} & J_0^{23} \\ 0 & J_0^{23} & J_0^{33} \end{Vmatrix}, \quad [\mathbf{J}_{2k}]_F = j_k\begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{Vmatrix}$$

$$[\mathbf{J}_{2k-1}]_F = j_k\begin{Vmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{Vmatrix}, \quad [\mathbf{J}_{2k-1,2k-1}]_F = j_{kk}\begin{Vmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix} \tag{1.6}$$

$$[\mathbf{J}_{2k,2k-1}]_F = [\mathbf{J}_{2k-1,2k}] = j_{kk}\begin{Vmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix}$$

$$j_k \equiv \int_0^1 \rho(a+s)\varphi_k ds, \quad j_{kk} \equiv \int_0^1 \rho\varphi_k^2 ds, \quad k = 1, 2, \dots$$

We will assume that, when rotated by an angle Δ about the O_1x_1 axis, the principal axes of inertia of the gyrostat without the rod with respect to the point O_1 transfer respectively into axes with unit vectors $\{\mathbf{f}_i\}$, $(i = 1, 2, 3)$. Then

$$\begin{aligned} J_0^{11} &= I_1^K + I_C, \quad J_0^{22} = I_2^K + (I_3^K - I_2^K)\sin^2\Delta, \quad J_0^{12} = J_0^{13} = 0 \\ J_0^{23} &= (I_3^K - I_2^K)\sin\Delta\cos\Delta, \quad J_0^{33} = I_3^K + I_C^3 - (I_3^K - I_2^K)\sin^2\Delta \end{aligned} \tag{1.7}$$

$I_C \equiv \int_0^1 \rho\ (a+s)^2 ds$ and I_C^3 is the moment of inertia of the undeformed rod about the axis O_1x_1 and its own undeformed axis respectively; here, the quantities I_j^K are the moments of inertia of the gyrostat housing about the axes O_1x_j $(j = 1, 2, 3)$. Note that all the other matrices of the components $[\mathbf{J}_{2k-1,p}]_F$ and $[\mathbf{J}_{2k,p}]_F$ are zero matrices by virtue of the properties of the functions $\{\chi_n\}$ $(k, p, n = 1, 2, \dots)$.

4. The central ellipsoid of inertia of the gyrostat and of the whole of the undeformed system is not an ellipsoid of revolution; similar assumptions have previously been used in [12–15].

It is well known (see [9], for example) that the equations of motion (various methods for obtaining these in the case of complex mechanical systems have been discussed in [15]) in the case being considered here, admit of, in addition to integrals of the direction cosines U_i $(i = 1, 2, 3)$, an integral of the Jacobi type U. We have

$$\begin{aligned} &U_1 \equiv \boldsymbol{\gamma}\boldsymbol{\gamma} - 1 = 0, \quad U_2 \equiv \boldsymbol{\beta}\boldsymbol{\beta} - 1 = 0, \quad U_3 \equiv \boldsymbol{\gamma}\boldsymbol{\beta} = 0 \\ &U \equiv T_r + \Pi + \Pi_g - \frac{1}{2}\boldsymbol{\omega}\mathbf{J}\boldsymbol{\omega} - \boldsymbol{\omega}\mathbf{k} = \text{const} \end{aligned} \tag{1.8}$$

Here $\mathbf{k}$ is the constant gyrostatic moment of the system, and the kinetic energy is defined by the expression

$$T_r \equiv \frac{1}{2}\boldsymbol{\Omega}\mathbf{J}\boldsymbol{\Omega} + \boldsymbol{\Omega}\mathbf{G} + \frac{1}{2}\sum_{n,m=1}^{\infty} a_{nm}\dot{q}_n\dot{q}_m$$

where $a_{nm} = \int_0^1 \rho\,\varphi_n\varphi_m ds$ and the angular momentum vector with respect to the point O has the form

$$\mathbf{G} \equiv \int_{\upsilon_2} (\mathbf{r} + \mathbf{u}) \times \dot{\mathbf{u}} dm = \sum_{n=1}^{\infty} \mathbf{G}_n\dot{q}_n + \sum_{n,p=1}^{\infty} \dot{q}_n q_p \mathbf{G}_{np}$$

The expressions for $\mathbf{G}_n$ and $\mathbf{G}_{np}$ as well as the equations of motion of the system with respect to its instantaneous centre of mass O are not presented here (see [15], for example). A partial derivative with respect to time is denoted by a dot.

2. FAMILIES OF EQUILIBRIA

In order to find the relative equilibria of the system and investigate their stability, we make us of the Routh–Lyapunov theorem [16, 11, 14] which is contained within the framework of the direct Lyapunov method [17–19]. We include the functionals V and V_1 using the formulae

$$V_1(\boldsymbol{\gamma}, \boldsymbol{\beta}, q, \lambda, \sigma, \nu) = \left(\Pi + \Pi_g - \frac{1}{2}\boldsymbol{\omega}\mathbf{J}\boldsymbol{\omega} - \mathbf{k}\boldsymbol{\omega}\right) + 3\omega^2\lambda U_3 + \frac{1}{2}\omega^2\nu U_2 - \frac{3}{2}\omega^2\sigma U_1$$

$$V(\boldsymbol{\Omega}, \dot{q}, \boldsymbol{\gamma}, \boldsymbol{\beta}, q, \lambda, \sigma, \nu) = T_r + V_1$$

where λ, σ and ν are undetermined Lagrange multipliers and the quantity in brackets on the right-hand side of the expression for V_1 is called the change in the potential energy. The derivative of the functional V with respective to time, that is, of the bunch of integrals of motion, is equal to zero.

Suppose the variables with a circumflex determine a certain unperturbed motion of the system (relative equilibrium when $\hat{\mathbf{\Omega}} = 0, \hat{q}^{\cdot} = 0$) and we denote small perturbations of the corresponding quantities by $\delta\mathbf{\Omega}$, $\delta\dot{q} \equiv (\delta\dot{q}_1, \delta\dot{q}_2, \ldots)^T$, $\delta q = (\delta q_1, \delta q_2, \ldots)^T$, $\delta\boldsymbol{\gamma}$, $\delta\boldsymbol{\beta}$. In a small neighbourhood of the unperturbed motion

$$W \equiv W(*) - V(0) = \delta V(0) + \delta^2 V(0) + Q(0)$$

where $V(*)$ and other quantities with the argument (*) are the values of the functional V and of the other quantities for the unperturbed motion, $V(0)$, $\delta V(0)$, $\delta^2 V(0)$ are the values of the functional V and its first and second variations and so on, calculated for the unperturbed motion of the system, and $Q(0)$ is a functional containing quantities of no less than the third order in the perturbations. If the unperturbed motion of the system is a relative equilibrium and it is precisely such equilibria that we shall consider below, it is possible to write the following equality

$$W \equiv V(*) - V(0) = T_r(*) + V_1(*) - T_r(0) - V_1(0) = T_r(*) + \delta V_1(0) + \delta^2 V_1(0) + Q(0) \qquad (2.1)$$

Under corresponding conditions [11] (in the case of a gyrostat with an elastic rod, for example when $\dot{q}(t) \in l_2$, $q(t) \in l_2$, $\{\Lambda_n^{-1}\} \in l_2$ and an orthonormalized system of functions $\{\chi_n\}$ [14], it can be shown that

$$\exists c_T > 0 : T_r > c_T\left(\mathbf{\Omega\Omega} + \sum_{n=1}^{\infty} \dot{q}_n^2\right)$$

The equations for finding the relative equilibria of the system and the undetermined Lagrange multipliers, which are written from the condition $\delta V(0) = 0$ ($\delta V(0) = \delta V_1(0)$ when $\hat{\mathbf{\Omega}} = 0, \hat{q}^{\cdot} = 0$), can be expressed as follows:

$$(\mathbf{J}(\hat{q}) - \sigma E)\hat{\boldsymbol{\gamma}} + \lambda\hat{\boldsymbol{\beta}} = 0 \Leftrightarrow \sigma = \hat{\boldsymbol{\gamma}}\mathbf{J}(\hat{\mathbf{q}})\hat{\boldsymbol{\gamma}}, \quad \lambda = -\hat{\boldsymbol{\beta}}\mathbf{J}(\hat{q})\hat{\boldsymbol{\gamma}}, \quad \hat{\boldsymbol{\alpha}}\mathbf{J}(\hat{q})\hat{\boldsymbol{\gamma}} = 0 \qquad (2.2)$$

$$(\nu E - \mathbf{J}(\hat{q}))\hat{\boldsymbol{\beta}} + 3\lambda\hat{\boldsymbol{\gamma}} - \boldsymbol{\eta} = 0 \Leftrightarrow \nu = \hat{\boldsymbol{\beta}}\mathbf{J}(\hat{q})\hat{\boldsymbol{\beta}} + \hat{\boldsymbol{\beta}}\boldsymbol{\eta}, \ \hat{\boldsymbol{\alpha}}\mathbf{J}(\hat{q})\hat{\boldsymbol{\beta}} = -\boldsymbol{\eta}\hat{\boldsymbol{\alpha}}, \ \hat{\boldsymbol{\gamma}}\mathbf{J}(\hat{q})\hat{\boldsymbol{\beta}} = -\boldsymbol{\eta}\hat{\boldsymbol{\gamma}}/4 \qquad (2.3)$$

$$\hat{q}_n + \omega^2(3\hat{\boldsymbol{\gamma}}\mathbf{J}'_n(\hat{q})\hat{\boldsymbol{\gamma}} - \mathrm{tr}\mathbf{J}'_n(\hat{q}) - \hat{\boldsymbol{\beta}}\mathbf{J}'_n(\hat{q})\hat{\boldsymbol{\beta}})/2 = 0, \quad n = 1, 2, \ldots \qquad (2.4)$$

The notation

$$\boldsymbol{\eta} \equiv \omega^{-1}\mathbf{k}, \quad \mathbf{J}'_n = \mathbf{J}_n + 2\sum_{p=1}^{\infty} \hat{q}_p \mathbf{J}_{np}$$

is used here.

By direct calculations using expressions (1.6) and the properties of the system of functions $\{\chi_n(s)\}$, we find that Eqs (2.2)–(2.4) admit of two single-parameter families of solutions, that is, of relative equilibria of a gyrostat with an elastic rod, which determine the corresponding non-trivial equilibrium orientations of the system to the attracting centre.

The first family (equilibria of the second class in accordance with the well-known classification [3, 8]) is characterized by the fact that the straight line, lying in the Ox_2x_3 plane and making an angle Θ_1 (the parameter of the family) with the unit vector $\mathbf{f}_3$, is directed towards the attracting centre. In projections onto the $\{\mathbf{f}_k\}$ axes, this family is defined by the following equations ($k = 1, 2, \ldots$)

$$\hat{\alpha}_1 = \pm 1, \quad \hat{\alpha}_2 = 0, \quad \hat{\alpha}_3 = 0; \quad \hat{\beta}_1 = 0, \quad \hat{\beta}_2 = \cos\Theta_1, \quad \hat{\beta}_3 = \sin\Theta_1 \qquad (2.5)$$
$$\hat{\gamma}_1 = 0, \quad \hat{\gamma}_2 = -\hat{\alpha}_1\sin\Theta_1, \quad \hat{\gamma}_3 = \hat{\alpha}_1\cos\Theta_1$$

$$\hat{q}_{2k-1} = 0, \quad \hat{q}_{2k} = \frac{\omega^2\sin\Theta_1\cos\Theta_1}{2[1 + \omega^2\Lambda_k^{-2}(1 - 4\sin^2\Theta_1)]\Lambda_k}\int_0^1 \rho(a+s)\chi_k ds \qquad (2.6)$$

Note that the denominator of the fraction does not vanish on account of the smallness of ω and $\Lambda_k > 1$, and that $\{\hat{q}_{2k}\} \in l_2$ if $\{\Lambda_k^{-1}\} \in l_2$. Actually, on taking account of the properties of Λ_k and $\{\chi_k\}$ from equalities (2.6) we derive the simple estimates

$$|\hat{q}_{2k}| < \frac{\omega^2}{2(1-3\omega^2\Lambda_1^{-2})\Lambda_k}\left(\int_0^1 \rho(a+s)^2 ds\right)^{1/2}\left(\int_0^1 \rho\chi_k^2 ds\right)^{1/2} = \frac{\omega^2\Lambda_1^2}{2(\Lambda_1^2-3\omega^2)\Lambda_k} I_C^{1/2}$$

$$\sum_{k=1}^{\infty} \hat{q}_{2k}^2 < \left(\frac{\omega^2\Lambda_1^2}{2(\Lambda_1^2-3\omega^2)}\right)^2 I_c \sum_{k=1}^{\infty} \Lambda_k^2$$

We assume here that the condition imposed on the lowest frequency: $\Lambda_1^2 > 3\omega^2$, which is often cited in papers on the stability of complex mechanical systems [9], is satisfied.

For this family of equilibria, the matrix of the components of the inertia tensor of the system has the form

$$[\mathbf{J}(\hat{q})]_F = (J_{ij}^F(\hat{q})) = \begin{Vmatrix} J_0^{11} + \sum_1 & 0 & 0 \\ 0 & J_0^{22} & J_0^{23} - \sum_2 \\ 0 & J_0^{23} - \sum_2 & J_0^{33} + \sum_1 \end{Vmatrix} \tag{2.7}$$

where

$$\sum_1 \equiv \sum_k \hat{q}_{2k}^2 j_{kk}, \quad \sum_2 \equiv \sum_k \hat{q}_{2k}^2 j_k$$

Obviously, the axes with the unit vectors $\{\mathbf{e}_k\}$, $\mathbf{e}_1 = \mathbf{f}_1$ in which the matrix $[\mathbf{J}(\hat{q})]_E = (J_{ii}^E)$ is diagonal are simply determined. Naturally, it is necessary in this case to convert the components of the vectors $\hat{\boldsymbol{\alpha}}$, $\hat{\boldsymbol{\beta}}$, $\hat{\boldsymbol{\gamma}}$ using the corresponding transformation matrix.

In accordance with formulae (2.2), the undetermined Lagrange multipliers are expressed in the form

$$\begin{aligned} \lambda(0) &= \hat{\alpha}_1(J_{22}^F - J_{33}^F)\sin\Theta_1\cos\Theta_1 - \hat{\alpha}_1 J_{23}^F(\cos^2\Theta_1 - \sin^2\Theta_1) \\ \sigma(0) &= J_{22}^F\sin^2\Theta_1 - 2J_{23}^F\sin\Theta_1\cos\Theta_1 + J_{33}^F\cos^2\Theta_1 \end{aligned} \tag{2.8}$$

In this case, as is also typical for the inverse problem of the equilibrium of a gyrostat without an elastic element, the parameter υ remains undetermined since the gyrostatic moment and, more precisely, its projection onto the normal to the orbital plane, has not been fixed (see (2.3)). Below, we shall make use of the choice of υ in order to ensure the stability of the equilibria of a gyrostat with an elastic rod.

The components of the vector of the gyrostatic moment $\mathbf{k}$ in the axes $\{\mathbf{f}_k\}$ for realizing the equilibria (2.5), (2.6) taking account of expressions (2.8) must be determined by the following equalities which are obtained from Eqs (2.3)

$$\begin{aligned} k_1 &= 0, \quad \omega^{-1}k_2 = (\nu - J_{22}^F(\hat{q}))\cos\Theta_1 - (3\lambda\hat{\alpha}_1 + J_{23}^F(\hat{q}))\sin\Theta_1 \\ \omega^{-1}k_3 &= (3\lambda\hat{\alpha}_1 - J_{23}^F(\hat{q}))\cos\Theta_1 + (\nu - J_{33}^F(\hat{q}))\sin\Theta_1 \end{aligned} \tag{2.9}$$

The second single parameter family (equilibria of the third class in accordance with the well-known classification [3, 8]) is characterized by the fact that the Ox_1 axis is directed towards the attracting centre (or from it) and, at the same time, the normal to the orbital plane makes an angle Θ_2 (the parameter of the family) with the unit vector $\mathbf{f}_2$. In the axes $\{\mathbf{f}_k\}$, this family of equilibria is defined by the equations

$$\begin{aligned} &\hat{\gamma}_1 = \pm 1, \quad \hat{\gamma}_2 = 0, \quad \hat{\gamma}_3 = 0; \quad \hat{\beta}_1 = 0, \quad \hat{\beta}_2 = \cos\Theta_2, \quad \hat{\beta}_3 = \sin\Theta_2 \\ &\hat{\alpha}_1 = 0, \quad \hat{\alpha}_2 = \hat{\gamma}_1\sin\Theta_2, \quad \hat{\alpha}_3 = -\hat{\gamma}_1\cos\Theta_2 \end{aligned} \tag{2.10}$$

$$\hat{q}_{2k-1} = 0, \quad \hat{q}_{2k} = -\frac{\omega^2\sin\Theta_2\cos\Theta_2}{[1+\omega^2\Lambda_k^{-2}(2+\cos^2\Theta_2)]\Lambda_k}\int_0^1 \rho(a+s)\chi_k ds; \quad k = 1, 2, \dots \tag{2.11}$$

The components of the inertia tensor of the system with respect to the axes $\{\mathbf{f}_k\}$ will, as previously, be given by formula (2.7), but taking Eqs (2.10) and (2.11) into account. In this case, the undetermined Lagrange multipliers have the form

$$\lambda(0) = 0, \quad \sigma(0) = J_{11}^F(\hat{q}) \tag{2.12}$$

Conclusions, similar to those above, hold with regard to the multiplier ν which is determined in accordance with the equalities (2.3) and (2.10)–(2.12). The components of the vector of the gyrostatic moment of the system in the axes $\{\mathbf{f}_k\}$ which ensures the equilibria (2.10), (2.11) are given by equalities which follow from expressions (2.3)

$$\begin{aligned} &k_1 = 0, \quad \omega^{-1}k_2 = (\nu - J_{22}^F(\hat{q}))\cos\Theta_2 - J_{23}^F(\hat{q})\sin\Theta_2 \\ &\omega^{-1}k_3 = -J_{23}^F(\hat{q})\cos\Theta_2 + (\nu - J_{33}^F(\hat{q}))\sin\Theta_2 \end{aligned} \tag{2.13}$$

The estimates

$$|\hat{q}_{2k}| < \frac{\omega^2}{\Lambda_k}\left|\int_0^1 \rho(a+s)\chi_k ds\right| < \frac{\omega^2}{\Lambda_k} I_C^{1/2} < \frac{\omega^2}{\Lambda_1} I_C^{1/2}$$

$$\sum_k \hat{q}_{2k}^2 < \omega^4 I_C \sum_k \Lambda_k^{-2}$$

hold for the values of (2.11)

3. STABILITY OF THE FAMILIES OF EQUILIBRIA

The conditions for the stability of the families of equilibria of the system which have been found in accordance with the Routh–Lyapunov theorem and the method for finding the minimum of a functional W will be obtained as conditions for the second variation of the functional V_1 (see (2.1)), calculated for the corresponding equilibria, to be positive definite. We now introduce the quantities

$$w_1 \equiv \delta\gamma_1, \quad w_2 \equiv \delta\beta_1, \quad w_3 \equiv \delta\gamma_2, \quad w_4 \equiv \delta\beta_2, \quad w_5 \equiv \delta\gamma_3, \quad w_6 \equiv \delta\beta_3$$
$$w = (w_1, \ldots, w_6), \quad \delta q = (\delta q_1, \ldots, \delta q_n, \ldots)$$

We assume that $(w, \delta q) \in l_2$, and we represent the second variation as follows [14]:

$$\delta^2 V_1(0) = \omega^2 (w, \delta q)\begin{Vmatrix} A & B \\ B^T & C \end{Vmatrix}(w, \delta q)^T$$

where, subject to the condition that the corresponding tensors and vectors are given by their own components in the axes $\{\mathbf{f}_k\}$, the matrix $A \equiv (A_{ij})$ $(i, j = 1, \ldots, 6; k = 1, 2, 3)$ only has the following non-zero components

$$A_{2k-1,2k-1} = 3(J_{kk}^F - \sigma(0)), \quad A_{2k,2k} = (\nu - J_{kk}^F)$$
$$A_{12} = A_{21} = A_{34} = A_{43} = A_{56} = A_{65} = 3\lambda(0)$$
$$A_{35} = 3J_{23}^F = A_{53}, \quad A_{46} = J_{23}^F = A_{64}$$

The matrix B has six infinite rows $b_i = (b_{i1}, b_{i2}, \ldots)$ $(i = 1, \ldots, 6; k, p = 1, 2, \ldots)$, the elements of which are determined using the formulae (taking into account that $\hat{q}_{2k-1} = 0$)

$$\begin{Vmatrix} b_{1,2k-l} \\ b_{3,2k-l} \\ b_{5,2k-l} \end{Vmatrix} = 3(\mathbf{J}_{2k-l} + \hat{q}_{2k}\mathbf{J}_{2k-l,2k})\hat{\boldsymbol{\gamma}}, \quad \begin{Vmatrix} b_{2,2k-l} \\ b_{4,2k-l} \\ b_{6,2k-l} \end{Vmatrix} = -(\mathbf{J}_{2k-l} + \hat{q}_{2k}\mathbf{J}_{2k-l,2k})\hat{\boldsymbol{\beta}}; \quad l = 0, 1 \tag{3.1}$$

The matrix C, with which we shall also identify a linear operator C, defined in l_2, with values, generally speaking, in the space S of infinite sequences, is, in this case, a partitioned diagonal matrix (with 2×2 blocks)

$$C = \mathrm{diag}(C_1, C_2, \ldots), \quad C_k = (C_{ij}^k)$$
$$C_{ii}^k = \omega^{-2} + 3\hat{\boldsymbol{\gamma}}\mathbf{J}_{2k-2+i,\,2k-2+i}\hat{\boldsymbol{\gamma}} - \mathrm{tr}\mathbf{J}_{2k-2+i,\,2k-2+i} - \hat{\boldsymbol{\beta}}\mathbf{J}_{2k-2+i,\,2k-2+i}\hat{\boldsymbol{\beta}}$$
$$C_{12}^k = C_{21}^k = (3\hat{\boldsymbol{\gamma}}\mathbf{J}_{2k-1,\,2k}\hat{\boldsymbol{\gamma}} - \mathrm{tr}\mathbf{J}_{2k-1,\,2k} - \hat{\boldsymbol{\beta}}\mathbf{J}_{2k-1,\,2k}\hat{\boldsymbol{\beta}})/2; \quad i, j = 1, 2; \quad k = 1, 2, \ldots$$

It should also be kept in mind that linear relations, arising from the conditions $\delta U_i(0) = 0$ ($i = 1, 2, 3$), are imposed on the quantities w and, in fact

$$\delta U_1(0) \equiv \hat{\gamma}_1 w_1 + \hat{\gamma}_2 w_3 + \hat{\gamma}_3 w_5 = 0, \quad \delta U_2(0) \equiv \hat{\beta}_1 w_2 + \hat{\beta}_2 w_4 + \hat{\beta}_3 w_6 = 0$$
$$\delta U_3(0) \equiv \hat{\beta}_1 w_1 + \hat{\gamma}_1 w_2 + \hat{\beta}_2 w_3 + \hat{\gamma}_2 w_4 + \hat{\beta}_3 w_5 + \hat{\gamma}_3 w_6 = 0 \tag{3.2}$$

The elements of the matrix C are given by the expressions ($k = 1, 2, \ldots$):
for the family of equilibria (2.5)–(2.9)

$$C_{11}^k = \omega^{-2}, \quad C_{12}^k = C_{21}^k = 0, \quad C_{22}^k = \omega^{-2} + \Lambda_k^{-2}(4\cos^2\Theta_1 - 3)$$

and for the family of equilibria (2.10)–(2.13), (2.7)

$$C_{11}^k = \omega^{-2} - 3\Lambda_k^{-2}, \quad C_{12}^k = C_{21}^k = 0, \quad C_{22}^k = \omega^{-2} + \Lambda_k^{-2}\cos^2\Theta_2$$

It clearly follows from the above expressions that the condition for the matrix C to be positive definite reduce to the condition $\Lambda_1^2 > 3\omega^2$, and it is possible to write

$$\exists\varepsilon_c > 0 : \Lambda_1^2 > 3\omega^2 + \varepsilon_C, \quad qCq^T > \varepsilon_C\|q\|^2 \Rightarrow \exists C^{-1}, \quad \|C^{-1}\| < \varepsilon_C^{-1} \tag{3.3}$$

The implication in expression (3.3) is true by virtue of the well-known theorem of analysis concerning an inverse operator (see [20], for example). In this case, it is possible to give an explicit representation for C^{-1} (subject to the condition that $\Lambda_1^2 > 3\omega^2$) and it is possible to determine $C^{-1/2}$ but only an estimate of the norm $\|C^{-1}\|$ is used in the following discussion. Now, when conditions (3.3) are satisfied, the following expression holds

$$\omega^{-2}\delta^2 V_1(0) = w(A - \varepsilon^{-2}BC^{-1}B^T)w^T + (1 - \varepsilon^2)\delta qC\delta q^T +$$
$$+ (\varepsilon^{-2}wBC^{-1} + \delta q)\varepsilon^2 C(\delta q^T + \varepsilon^{-2}C^{-1}B^T w^T)$$

This is easily verified by direct calculation in the case of arbitrary $\varepsilon \in (0, 1)$. It can be seen that the conditions for $\delta^2 V_1(0)$ to be positive definite can be obtained as conditions for a quadratic form with a matrix $(A - \varepsilon^{-2}B\,C^{-1}B^T)$ to be positive definite when the linear relations (3.2) exist.

We will now introduce the quantities

$$d_{ij} \equiv \varepsilon^{-2}b_i C^{-1}b_j^T = d_i^T d_j, \quad d_i = \varepsilon^{-1}C^{-1/2}b_i^T; \quad i, j = 1, \ldots, 6$$

for which the following limits exist

$$|d_{ij}| \le \varepsilon^{-2}\varepsilon_C^{-1}\|b_i\|\|b_j\|$$

Using the Cauchy–Bunyakovskii inequality, it can be shown that $\{b_i\} \in l_2$ when $\{\Lambda_n^{-1}\} \in l_2$.

Depending on the choice of the system of coordinates, the expressions for d_{ij}, as well as for b_i (see (3.1)), will change. Generally speaking, the system of coordinates (the axes $\{\mathbf{e}_k\}$ or $\{\mathbf{f}_k\}$) using which d_{ij}, b_i etc. are calculated, and it is important to note this, will be indicated with the corresponding superscript d_{ij}^F or b_i^E, etc. This conversion is also used in the case of the parameters for the families of equilibria Θ_1 and Θ_2. Note that the expressions for the elements of the matrix C do not change with the choice of the axes and, when the axes $\{\mathbf{e}_k\}$ are used, there will naturally be zero elements among the elements of the matrix $A = (A_{ij}^E)$: $A_{35}^E = A_{53}^E = 0, A_{46}^E = A_{64}^E = 0$, unlike the values of these elements when the axes $\{\mathbf{f}_k\}$ are used (see above).

Omitting the lengthy intermediate calculations in accordance with the method for investigating that the quadratic form of a finite number of variables is positive definite when there are linear relations which, in the case under consideration, leads to requirements that the corresponding determinants of the seventh, eighth and ninth orders [19, 21] ($\Delta_7(0) > 0, \Delta_8(0) > 0, \Delta_9(0) > 0$) are positive, we formulate an assertion concerning the stability of the non-trivial, single-parameter families of equilibria which have been found.

Assertion 1. It is sufficient to satisfy the following conditions for the family (2.5)–(2.9) of uniaxial, non-trivial orientations of a gyrostat with an elastic rod toward an attracting centre to be stable.

$$\Lambda_1^2 > 3\omega^2, \quad \{\Lambda_n^{-1}\} \in l_2 \tag{3.4}$$

$$J_{11}^E > I_{22}^E \sin^2\Theta_1^E + I_{33}^E \cos^2\Theta_1^E - \frac{1}{3} d_{11}^E \Leftrightarrow 3(J_{11}^E - \sigma) - d_{11}^E > 0 \tag{3.5}$$

$$\begin{aligned} \nu > {} & J_{33}^E(\hat\beta_2^E)^2 + J_{22}^E(\hat\beta_3^E)^2 - 3(J_{33}^E - \sigma)(\hat\gamma_2^E)^2 - 3(J_{22}^E - \sigma)(\hat\gamma_3^E)^2 + \\ & + (d_{16}^E \hat\beta_2^E + d_{15}^E \hat\gamma_2^E - d_{14}^E \hat\beta_3^E - d_{13}^E \hat\gamma_3^E)^2 / (3(J_{11}^E - \sigma) - d_{11}^E) \end{aligned} \tag{3.6}$$

$$\nu > \nu_2 \tag{3.7}$$

where ν_2 is the larger root of the quadratic equation in υ which is obtained from the condition that $\Delta_9(0) = 0$ (the expression for ν_2 is not written out because of its length).

Note that conditions (3.50 and (3.6) are more rigorous compared with the analogous conditions which are used for a mechanical system which has "solidified" in the equilibrium being considered, on account of the existence of the term $d_{11} \geq 0$ and, correspondingly, on account of the presence of a non-negative last term in inequality (3.6). Condition (3.5), which is independent of the parameter $\nu \equiv \hat{\boldsymbol\beta}\mathbf{J}(0)\hat{\boldsymbol\beta} + \omega^{-1}\mathbf{k}\hat{\boldsymbol\beta}$, is clearly not satisfied if $J_{11}^E = \min_k J_{kk}^E$ in the case of the equilibrium orientation being considered, that is, if the ellipsoid of inertia of the system in equilibrium with the major axis is directed along the tangent to the orbit. However, this condition can be satisfied by an appropriate choice of the moments of inertia of the gyrostat. In fact, the following chain of inequalities can be shown to hold on the basis of formulae (2.6)–(2.8), the Cauchy – Bunyakovskii inequality and the estimate for d_{11}

$$\begin{aligned} & 3(J_{11}^E - \sigma) - d_{11}^E > 0 \Rightarrow J_{11}^F - J_{22}^F \sin\Theta_1 + 2J_{23}^F \sin\Theta_1 \cos\Theta_1 - J_{33}^F \cos^2\Theta_1 - d_{11}^F/3 > 0 \Leftrightarrow \\ & \Leftrightarrow (I_1^K - I_2^K)\sin^2\Theta_1 + (I_1^K - I_3^K)\cos^2\Theta_1 + (I_3^K - I_2^K)\sin\Delta\sin(\Delta + 2\Theta_1) + \\ & + \left(\int \rho(a+s)^2 ds\right)\cos^2\Theta_1 + \sin^2\Theta_1\left(\sum_k \hat q_{2k}^2 \int \rho\varphi_k^2 ds\right) - \\ & - \sin 2\Theta_1\left(\sum_k \hat q_{2k} \int \rho(a+s)\varphi_k ds\right) - d_{11}^F/3 \Leftarrow \\ & \Leftarrow (I_1^K - I_2^K)\sin^2\Theta_1 + (I_1^K - I_3^K)\cos^2\Theta_1 + (I_3^K - I_2^K)\sin\Delta\sin(\Delta + 2\Theta_1) - \\ & - \sum_k \hat q_{2k}^2 - \int \rho(a+s)^2 ds \sum_k \int \rho\varphi_k^2 ds - d_{11}^F/3 \Leftarrow \\ & \Leftarrow (I_1^K - I_l^K) - (I_l^K - I_m^K) - \sum_k \hat q_{2k}^2 - I_C \sum_k \Lambda_k^{-2} - 3\varepsilon^2 \varepsilon_C^{-1} \Lambda_1^{-4} \sum_k \hat q_{2k}^2 > 0 \end{aligned}$$

In the last inequality, we assume that $I_1^K > I_l^K > I_m^K$, $l, m = 2, 3$; $l \neq m$ and, taking into account the estimate for $\sum \hat{q}_{2k}^2$, we conclude that it and, consequently, inequality (3.5) are satisfied if

$$I_1^K - I_l^K > I_l^K - I_m^K + \left((1 + 3\varepsilon^{-2}\varepsilon_C^{-1})\left(\frac{\omega^2 \Lambda_1^2}{2(\Lambda_1^2 - 3\omega^2)}\right)^2 + 1\right) I_C \sum_k \Lambda_k^{-2} \tag{3.8}$$

It is clear from this that, in order for conditions (3.5) to be satisfied, it is sufficient that the difference between the larger and average moments of inertia of the gyrostat should be greater than the difference between its average and the smaller moment of inertia by an amount which is determined by the second term on the right-hand side of inequality (3.8). This amount depends on the geometric, mass and stiffness characteristics of the elastic rod. If the inequality $I_1^K > I_l^K > I_m^K$ is taken into account, it follows from expression (3.8) that the smaller moment of inertia of the gyrostat I_m^K is also greater than the difference between its average and smaller moments of inertia by the same amount.

It can be shown that condition (3.5) is satisfied when $J_{11}^E = \underset{i}{\operatorname{mid}} J_{ii}^E$ if the vector $\hat{\boldsymbol{\gamma}}$ is such that the vector $\mathbf{J}^{1/2}\hat{\boldsymbol{\gamma}}$ lies in the "interior" of the domain between circular cross-sections of the central gyrational ellipsoid (constructed for the equilibrium of the system being considered) containing its minor axis ($J_{11}^E \neq J_{22}^E \neq J_{33}^E$). In the "interior" means that the length of the vector $\mathbf{J}^{1/2}\hat{\boldsymbol{\gamma}}$, the end of which is situated on the surface if the above-mentioned ellipsoid, is smaller than the quantity $(J_{11}^E - d_{11}^E/3)$, which obviously must be greater than J_{ii}^E. The gyrational ellipsoid of the system is linked to its ellipsoid of inertia [22]. Here and henceforth,

$$J_{11}^E = \underset{i}{\operatorname{mid}} J_{ii}^E \Leftrightarrow J_{ll}^E < J_{11}^E < J_{kk}^E (\{l, k\} = \{2, 3\},\ l \neq k)$$

Assertion 2. The family (2.10)–(2.13), (2.7) of uniaxial, non-trivial orientations of a gyrostat with an elastic rod toward an attracting centre will be stable if

$$\Lambda_1^2 > 3\omega^2, \quad \{\Lambda_n^{-1}\} \in l_2 \tag{3.9}$$

$$\upsilon \equiv 3(J_{33}^E - \sigma)(\hat{\alpha}_3^E)^2 + 3(J_{22}^E - \sigma)(\hat{\alpha}_2^E)^2 - (d_5^E \hat{\alpha}_3^E - d_3^E \hat{\alpha}_2^E)^2 > 0, \quad \sigma = J_{11}^E \tag{3.10}$$

$$\begin{aligned} \nu > {} & J_{33}^E(\hat{\alpha}_3^E)^2 + J_{22}^E(\hat{\alpha}_2^E)^2 + (d_4^E \hat{\beta}_3^E + d_6^E \hat{\beta}_2^E)^2 + \\ & + (d_{56}^E \hat{\alpha}_3^E \hat{\beta}_2^E - d_{34}^E \hat{\alpha}_2^E \hat{\beta}_3^E - d_{45}^E \hat{\alpha}_3^E \hat{\beta}_3^E + d_{36}^E \hat{\alpha}_2^E \hat{\beta}_2^E)^2 / \upsilon \end{aligned} \tag{3.11}$$

$$\nu > \nu_2 \tag{3.12}$$

where ν_2 is the larger root of the quadratic equation in ν obtained from the corresponding condition $\Delta_9(0) = 0$.

Arguments similar to those presented earlier in [19, p. 271] and based on the methods of the theory of bifurcations hold with respect to the existence of real roots $\nu_1 \leq \nu_2$ of the equation $\Delta_9(0) = 0$. Conditions (3.10) and (3.11) are more rigorous than the analogous conditions which can be obtained for a gyrostat without an elastic element which, as regards its inertia characteristics, is equivalent to the system under investigation which has been "solidified" in the corresponding equilibrium. Inequality (3.1) is obviously not satisfied if $J_{11}^E = \underset{i}{\max} J_{ii}^E$, that is, when the minor axis of the central ellipsoid of inertia of the system, constructed for the equilibrium being considered, is located along the radius vector of the orbit (along the unit vector $\hat{\boldsymbol{\gamma}}$). Condition (3.10) can be satisfied when $J_{11}^E = \underset{i}{\operatorname{mid}} J_{ii}^E$, if $\hat{\boldsymbol{\alpha}}$ is such that the vector $\mathbf{J}^{1/2}\hat{\boldsymbol{\alpha}}$, the end of which is located on the surface of the central gyrational ellipsoid of the system in the equilibrium being considered, lies in the "interior" of the domain between the circular cross-sections containing the major axis of the above-mentioned ellipsoid. In this case, in the "interior" means that the length of the vector $\mathbf{J}^{1/2}\hat{\boldsymbol{\alpha}}$ is greater than the quantity $J_{11}^E + (d_5^E \hat{\alpha}_3^E - d_3^E \hat{\alpha}_2^E)/3$ which, in its turn, must be smaller than $\underset{i}{\max} J_{ii}^E$, otherwise condition (3.10) is not satisfied for any values of $\hat{\boldsymbol{\alpha}}$.

By reasoning in a similar manner to that above in obtaining expression (3.8), it can be shown that condition (3.10) is satisfied regardless of the value of the parameter of the family Θ_2 and the deformations of the rod for an appropriate choice of the moments of inertia of the gyrostat.

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